

VECTOR FIELD MEDIATED MODELS OF DYNAMICAL LIGHT VELOCITY

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ABSTRACT. A vector-tensor theory of gravity that was introduced in an earlier publication is analyzed in detail and its consequences for early universe cosmology are examined. The multiple light cone structure of the theory generates different speeds of gravitational and matter wave fronts, and the contraction of these light cones produces acausal, superluminary inflation that can resolve the initial value problems of cosmology.

1. INTRODUCTION

In previous work [1, 2], we have introduced a new kind of vector-tensor and scalar-tensor theory of gravity, which exhibits a bimetric structure and contains two or more light cones. This type of model has attracted some interest recently [3, 4, 5], and similar effects have been noted elsewhere [6]. The motivation for considering these models is derived from earlier work of one of the present authors [7], which provided a scenario in which some of the outstanding issues in cosmology can be resolved. The present line of work provides a specific class of models that realize these ideas, for it provides a fundamental dynamical mechanism for varying speed of light theories and generates a new mechanism for an inflationary epoch that could solve the initial value problems [7] of early universe cosmology.

In this article, we focus on clarifying the role that these models can play in the early universe, demonstrating how matter that satisfies the strong energy condition can nevertheless contribute to the cosmic acceleration. Recently, an analysis of a similar class of theories has appeared [3, 4] which, while introducing some interesting ideas, unfortunately claimed that we had made an algebraic error in our previous work [1]. This can be attributed to a misunderstanding of part of the construction that was perhaps not dealt with in sufficient detail in our initial publication. We will rectify this situation here, developing the model in additional detail and showing explicitly that the error attributed to us in [3] is in fact an error on their part. We will also show that our cosmological model can be mapped to a model with varying fundamental constants [8, 9, 10], albeit not uniquely and requiring some care in the interpretation of the varying constants that appear.

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It is hoped that the models can shed some light on the new observational data that suggests the expansion of the universe at present is undergoing an acceleration [11, 12, 13]. Although there has been some success in understanding the latter problem by the inclusion of a class of very particular scalar field potentials [14], it is fair to say that not all issues have been resolved. In this article, we will not have much to say about this issue since, as we shall see, the vector field that produces a superluminary expansion in the early universe must vanish at some scale, and standard cosmology results afterwards. Using the scalar field version of the model, we expect that not only will we be able to generate sufficient inflation, but that a quintessence-like solution should be achievable. We shall concentrate our efforts on the vector-tensor model by providing a more detailed analysis of its consequences and postpone a fuller analysis of the scalar-tensor model to a future article.

2. THE VECTOR MEDIATED MODEL

We shall be considering models with an action of the form

$$S = \bar{S}_{\text{gr}}[\bar{g}] + S[g, \psi] + \hat{S}[\hat{g}, \hat{\phi}^I]. \quad (1)$$

The first term is the usual Einstein-Hilbert action for general relativity constructed from a metric $\bar{g}_{\mu\nu}$, and the final term is the contribution from the non-gravitational (matter) fields in spacetime $\hat{\phi}^I$, and is built from a different but related metric $\hat{g}_{\mu\nu}$. One motivation for constructing the action (1) is simplicity: it allows us to build models in a modular way with additional matter fields introduced as necessary, and it requires only fairly well-known variational results. The other benefit is that it makes fairly clear what metrics will be of physical relevance.

The contribution $S[g, \psi]$ is constructed from a metric $g_{\mu\nu}$ and includes kinetic terms for a field or fields (unspecified as yet) ψ that may be considered to be part of the gravitational sector, modifying the reaction of spacetime to the presence of the matter fields in $\hat{S}[\hat{g}, \hat{\phi}^I]$. The manner in which that ψ accomplishes this is by modifying the metric that appears in each of the actions. For example, in [1] ψ was a vector field, $\bar{g}_{\mu\nu} = g_{\mu\nu}$ and $\hat{g}_{\mu\nu} = g_{\mu\nu} + b\psi_\mu\psi_\nu$, whereas in [2] ψ was a scalar field, $\bar{g}_{\mu\nu} = g_{\mu\nu}$ and $\hat{g}_{\mu\nu} = g_{\mu\nu} + b\partial_\mu\psi\partial_\nu\psi$. These relations imply that matter and gravitational fields propagate at different velocities if ψ is nonvanishing. In this work we will explore the first of these possibilities in more detail than was possible in our previous publication [1], clarifying some misinterpretations that have appeared in recent work [3].

Since the matter action \hat{S} is built using only $\hat{g}_{\mu\nu}$, it is the null surfaces of $\hat{g}_{\mu\nu}$ along which matter fields propagate. If we assume that other than the presence of a “composite” metric the matter action is otherwise a conventional form (perfect fluid, scalar field, Maxwell, *etc.*), then variation of the

matter action

$$\delta \hat{S}[\hat{g}, \hat{\phi}^I] = \int d\hat{\mu} \left(-\hat{F}_I \delta \hat{\phi}^I - \frac{1}{2} \hat{T}^{\mu\nu} \delta \hat{g}_{\mu\nu} \right), \quad (2)$$

provides the matter energy-momentum tensor $\hat{T}^{\mu\nu}$, which will be conserved

$$\hat{\nabla}_\nu \hat{T}^{\mu\nu} = 0, \quad (3)$$

by virtue of the matter field equations $\hat{F}_I = 0$. Throughout we will write, for example, $\hat{\nabla}_\nu$ for the covariant derivative constructed from the Levi-Civita connection of $\hat{g}_{\mu\nu}$. Since we also assume that the matter fields satisfy the dominant energy condition, we therefore know (assuming appropriate smoothness of $\hat{g}_{\mu\nu}$) that matter fields cannot travel faster than the speed of light as determined by $\hat{g}_{\mu\nu}$ [15].

The gravitational action is written

$$\bar{S}_{\text{gr}}[\bar{g}] = -\frac{1}{\kappa} \int d\bar{\mu} \bar{R}, \quad (4)$$

where we use a metric with $(+---)$ signature and have defined $\kappa = 16\pi G/c^4$. We will denote the metric densities by, *e.g.*, $\bar{\mu} = \sqrt{-\det(\bar{g}_{\mu\nu})}$ and in addition write $d\bar{\mu} = \bar{\mu} dt d^3x$. The variation of (4) is

$$\delta \bar{S}_{\text{gr}}[\bar{g}] = \frac{1}{\kappa} \int d\bar{\mu} \bar{G}^{\mu\nu} \delta \bar{g}_{\mu\nu}. \quad (5)$$

We will not consider a cosmological constant—it is a trivial matter to include it later. Provided the resulting field equation for $\bar{G}^{\mu\nu}$ has nothing in it that disturbs the principal part of the field equations and the constraints remain constraints, then we can identify the metric $\bar{g}_{\mu\nu}$ as providing the light cone for the gravitational system.

It remains to connect these two pieces with specific models for S , $\hat{g}_{\mu\nu}$ and $\bar{g}_{\mu\nu}$. We will generalize and provide more details on the vector field model that we presented in [1]. The choice of this model over a scalar field mediated model is due to the fact that the vector field models do not have any remaining degrees of freedom in an FRW universe and are therefore simpler to analyze.

Keeping in mind the dangers involved in coupling vector fields to gravity [16, 17], we begin with a Proca model with arbitrary potential

$$S[g, \psi] = -\frac{1}{\kappa} \int d\mu \left(\frac{1}{4} B^2 - V(X) \right), \quad (6)$$

where we will use the definition

$$X = \frac{1}{2} \psi^2, \quad (7)$$

and $V'(X) = \partial V(X)/\partial X$. We will also use $B_{\mu\nu} = \partial_\mu \psi_\nu - \partial_\nu \psi_\mu$, $\psi^2 = g^{\mu\nu} \psi_\mu \psi_\nu$ and $B^2 = g^{\alpha\mu} g^{\beta\nu} B_{\alpha\beta} B_{\mu\nu}$. We will assume that as $\psi_\mu \rightarrow 0$ we have $V(X) \sim m^2 X$ and therefore the linearized (in ψ_μ) limit of our model is identical to Einstein-Proca field equations coupled to matter.

Using the standard energy-momentum tensor for the vector field

$$T^{\mu\nu} = -B^{\mu\alpha}B^\nu{}_\alpha + \frac{1}{4}g^{\mu\nu}B^2 + V'\psi^\mu\psi^\nu - Vg^{\mu\nu}, \quad (8)$$

the variation of (6) results in

$$\delta S[g, \psi] = \frac{1}{\kappa} \int d\mu \left(\nabla_\mu B^{\mu\alpha} + V'\psi^\alpha \right) \delta\psi_\alpha - \frac{1}{2\kappa} \int d\mu T^{\mu\nu} \delta g_{\mu\nu}. \quad (9)$$

Note that writing ψ^α is potentially ambiguous, since there is more than one way in which one can “raise” an index. We will use $\psi^\alpha = g^{\alpha\beta}\psi_\beta$, $\hat{\psi}^\alpha = \hat{g}^{\alpha\beta}\psi_\beta$ and $\bar{\psi}^\alpha = \bar{g}^{\alpha\beta}\psi_\beta$ as necessary, and consider ψ_μ as the independent components of the (co-)vector field. The same will hold for any other tensor, and we will be explicit about which contraction we are using where necessary.

In choosing the form of $\hat{g}_{\mu\nu}$ and $\bar{g}_{\mu\nu}$, we could be fairly general and define, for example

$$\hat{g}_{\mu\nu} = a(B^2, X)g_{\mu\nu} + b(B^2, X)\psi_\mu\psi_\nu + \text{other terms like } B^\alpha{}_\mu B_{\alpha\nu} \text{ etc.} \quad (10)$$

The presence of $B_{\mu\nu}$ in this relation would lead to second derivatives of ψ_μ appearing in the relationship between $\Gamma_{\mu\nu}^\alpha$ and $\hat{\Gamma}_{\mu\nu}^\alpha$, and therefore re-writing the field equations in terms of $\hat{g}_{\mu\nu}$ will have a nontrivial effect on the principal parts of the differential equations. What one needs is for ∇_μ and $\hat{\nabla}_\mu$ to differ only by first derivatives of ψ_μ , which is accomplished by considering the simpler form $\hat{g}_{\mu\nu} = a(X)(g_{\mu\nu} + b(X)\psi_\mu\psi_\nu)$. Although this more general class of models is worth considering, here we will limit ourselves to the choice

$$\hat{g}_{\mu\nu} = g_{\mu\nu} + b\psi_\mu\psi_\nu, \quad \bar{g}_{\mu\nu} = g_{\mu\nu} + g\psi_\mu\psi_\nu, \quad (11)$$

so that the variations of $\hat{g}_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ are related to those of $g_{\mu\nu}$ and ψ_μ by, for example

$$\delta\hat{g}_{\mu\nu} = \delta g_{\mu\nu} + b(\psi_\mu\delta\psi_\nu + \psi_\nu\delta\psi_\mu). \quad (12)$$

This type of model is therefore a “purely dynamical metric theory”, for all fields are determined by the coupled field equations, but the relationship (11) must be considered as ‘prior geometry’ [18]. The field equations

$$\nabla_\mu B^{\mu\nu} + V'\psi^\nu + gT^{\mu\nu}\psi_\mu + \kappa \frac{\hat{\mu}}{\mu} (g - b)\hat{T}^{\nu\mu}\psi_\mu = 0, \quad (13a)$$

$$\bar{\mu}\bar{G}^{\mu\nu} = \frac{1}{2}\mu T^{\mu\nu} + \frac{1}{2}\kappa\hat{\mu}\hat{T}^{\mu\nu}, \quad (13b)$$

result from assembling the contributions from (2), (5) and (9) proportional to $\delta\psi_\nu$ and $\delta g_{\mu\nu}$, respectively.

As we shall see from (15), making the local redefinition of metric fields (11) will not alter the principal part of any second-order field equation. We are therefore free to write the entire system (13) as a partial differential equation in terms of the fields $\bar{g}_{\mu\nu}$, ψ_μ and $\hat{\phi}^I$. While it is possible that constraints in the matter system $\hat{F}_I = 0$ will pick up terms containing first derivatives of ψ_μ , they will nonetheless remain constraints. It is therefore clear that

$\hat{g}_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ provide the characteristic surfaces for matter and gravitational fields, respectively. Unfortunately, although most of the construction of our model relies on the metric $g_{\mu\nu}$ and its variation in the action, it is not actually of any obvious physical relevance. This can be seen by considering the divergence of (13a) given in (18). When this is used to write the principal part of (13a) as a simple wave operator, we find that the derivatives of ψ_μ appearing in (18) make this impossible. The analysis of the characteristic speeds of such a wave operator are far from trivial (see, however, [18] for the speeds of cosmological perturbations).

3. THE BIANCHI IDENTITIES

A potential concern when considering the field equations (13b) is how the Bianchi identities $\bar{\nabla}_\nu \bar{G}^{\mu\nu} = 0$ and matter conservation laws (3) combine to ensure that the divergence of (13b) does not lead to any additional constraints on the system. Although it could be said that covariance guarantees this consistency, because we are not considering an explicit matter model, it is important to show the relationship between *any* matter model satisfying (3) and the Bianchi identities.

From the definitions (11) we find, for example

$$\hat{g}_{\mu\nu} := g_{\mu\nu} + b\psi_\mu\psi_\nu, \quad \hat{g}^{\mu\nu} = g^{\mu\nu} - \frac{b}{1+2bX}\psi^\mu\psi^\nu, \quad (14)$$

with similar relationships between the metrics $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ and the metrics $\hat{g}_{\mu\nu}$ and $\bar{g}_{\mu\nu}$, with suitable replacement of the constant b and re-definition of X . It is then a straightforward exercise to find

$$\hat{\Gamma}_{\mu\nu}^\alpha - \Gamma_{\mu\nu}^\alpha = \frac{b}{1+2bX}\psi^\alpha\nabla_{(\mu}\psi_{\nu)} - b\hat{g}^{\alpha\beta}B_{\beta(\mu}\psi_{\nu)}, \quad (15a)$$

$$\bar{\Gamma}_{\mu\nu}^\alpha - \Gamma_{\mu\nu}^\alpha = \frac{g}{1+2gX}\psi^\alpha\nabla_{(\mu}\psi_{\nu)} - g\bar{g}^{\alpha\beta}B_{\beta(\mu}\psi_{\nu)}, \quad (15b)$$

$$\bar{\Gamma}_{\mu\nu}^\alpha - \hat{\Gamma}_{\mu\nu}^\alpha = \left(\frac{g}{1+2gX} - \frac{b}{1+2bX}\right)\psi^\alpha\nabla_{(\mu}\psi_{\nu)} - (g\bar{g}^{\alpha\beta} - b\hat{g}^{\alpha\beta})B_{\beta(\mu}\psi_{\nu)}. \quad (15c)$$

Taking the divergence of (13b) and using the Bianchi identity $\bar{\nabla}_\nu \bar{G}^{\mu\nu} = 0$, results in

$$0 = \nabla_\nu[\mu T^{\mu\nu}] + \mu T^{\beta\alpha}(\bar{\Gamma}_{\alpha\beta}^\mu - \Gamma_{\alpha\beta}^\mu) + \kappa\hat{\mu}\hat{T}^{\beta\alpha}(\bar{\Gamma}_{\alpha\beta}^\mu - \hat{\Gamma}_{\alpha\beta}^\mu), \quad (16)$$

where we have re-written the covariant derivative $\bar{\nabla}_\nu$ in terms of ∇_ν when acting on the vector field energy-momentum tensor, and in terms of $\hat{\nabla}_\nu$ when acting on the matter energy-momentum tensor. We have also used (3) to drop the divergence that results from the latter. The goal here is to show that this equation is automatically satisfied, *i.e.*, it does not entail any additional restrictions on the matter fields or ψ_μ that are not already in existence in their respective field equations.

Taking the divergence of (8) and using $\nabla_\alpha B_{\beta\gamma} + \nabla_\beta B_{\gamma\alpha} + \nabla_\gamma B_{\alpha\beta} = 0$ and (13a), we find

$$\nabla_\nu[\mu T^{\mu\nu}] = \mu\psi^\mu\nabla_\nu[V'\psi^\nu] + (g\mu T^{\alpha\beta} + \kappa(g-b)\hat{\mu}\hat{T}^{\alpha\beta})B^\mu{}_\alpha\psi_\beta. \quad (17)$$

Now taking the divergence of (13a) and using (17) and (15), we find that

$$\begin{aligned} \mu(1+2gX)\nabla_\nu[V'\psi^\nu] + g\mu T^{\alpha\beta}(\nabla_\alpha\psi_\beta + g\psi_\alpha\psi^\mu B_{\mu\beta}) \\ + \kappa(g-b)\hat{\mu}\hat{T}^{\alpha\beta}(\hat{\nabla}_\alpha\psi_\beta + g\psi_\alpha\psi^\mu B_{\mu\beta}) = 0. \end{aligned} \quad (18)$$

Using this to replace $\nabla_\nu[V'\psi^\nu]$ in (17) and identifying the various connections from (15), we find that the right-hand side of (16) vanishes identically. Thus we know that *any* matter model that conserves energy-momentum with respect to $\hat{g}_{\mu\nu}$ is consistent with the gravitational structure that we have introduced.

4. CAUSALITY AND ENERGY CONDITIONS

The “most physical” metric is clearly $\hat{g}_{\mu\nu}$, since it describes the geometry on which matter propagates and interacts (*c.f.*, the argument following (3) and [15]). Because all matter fields are coupled to the same metric $\hat{g}_{\mu\nu}$ in exactly the same way, the weak equivalence principle is satisfied. Furthermore, because one can work in a local Lorentz frame of $\hat{g}_{\mu\nu}$, in which non-gravitational physics takes on its special relativistic form, the Einstein equivalence principle is also satisfied. However, because $\hat{g}_{\mu\nu}$ does *not* couple to matter in the same way as in general relativity unless $\psi_\mu = 0$, the strong equivalence principle will be violated. Clearly, we could introduce couplings of the form $\psi_\mu \hat{J}^\mu_{\text{matter}}$, where $\hat{J}^\mu_{\text{matter}}$ is a current constructed from the matter fields, into the action that would generically violate all equivalence principles.

In this type of model one cannot speak of causality without some qualification. From (11), $\hat{g}_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ are related by

$$\bar{g}_{\mu\nu} = \hat{g}_{\mu\nu} + (g-b)\psi_\mu\psi_\nu, \quad \bar{g}^{\mu\nu} = \hat{g}^{\mu\nu} - \frac{(g-b)}{1+(g-b)\hat{\psi}^2}\hat{\psi}^\mu\hat{\psi}^\nu. \quad (19)$$

Characteristic surfaces of the field equations are determined from each metric by $\bar{g}^{\alpha\beta}V_\alpha V_\beta = 0$ and $\hat{g}^{\alpha\beta}V_\alpha V_\beta = 0$, where V_α is a covector field that lies on the null cone of $\bar{g}^{\mu\nu}$ and $\hat{g}^{\mu\nu}$, respectively. Because these two metrics are generally different and define different characteristic surfaces, we must specify, for example, a \bar{g} -timelike vector V_μ as being one that satisfies $\bar{g}^{\alpha\beta}V_\alpha V_\beta > 0$. Because of this extended notion of causality, we will call a vector timelike if it is timelike with respect to all metrics in the theory, and spacelike if it is spacelike with respect to all the metrics in the theory [19].

Equation (19) allows us to connect these different causal relationships. Considering some covector V_μ , we have

$$\bar{g}^{\alpha\beta}V_\alpha V_\beta = \hat{g}^{\alpha\beta}V_\alpha V_\beta - \frac{(g-b)}{1+(g-b)\hat{\psi}^2}(\hat{g}^{\alpha\beta}V_\alpha\psi_\beta)^2, \quad (20)$$

and if we assume that V_μ is \hat{g} -null, then it is generically (assuming that $\hat{g}(\psi, V) \neq 0$, *i.e.*, ψ_μ and V_μ are not relatively \hat{g} -null)

$$\begin{cases} \bar{g} - \text{timelike} & b > g, \\ \bar{g} - \text{null} & b = g, \\ \bar{g} - \text{spacelike} & b < g. \end{cases} \quad (21)$$

The main motivation for considering these theories is that they should have something to say about the horizon problem in the early universe. If $\psi_\mu \neq 0$, then if we choose $b > g$, matter fields will propagate outside the light cone of the gravitational field as illustrated in Figure 1. As $\psi_\mu \rightarrow 0$ the

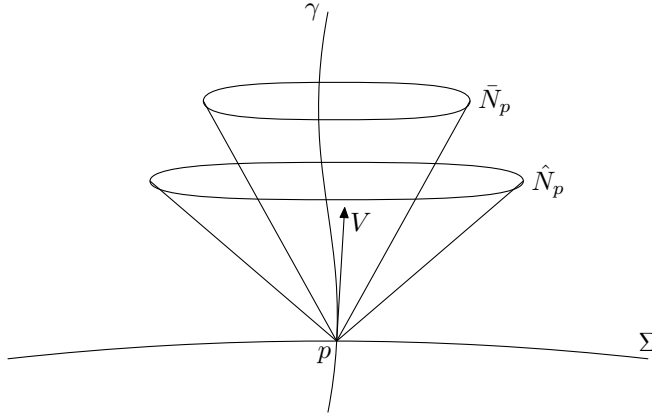


FIGURE 1. A timelike trajectory γ intersecting a spacelike hypersurface Σ at a point p . The cones \hat{N}_p and \bar{N}_p represent the null cones of $\hat{g}_{\mu\nu}$ and $\bar{g}_{\mu\nu}$, respectively. We have shown the case where $b > g$.

matter light cone will ‘contract’ and matter and gravitational disturbances will eventually propagate at the same velocity. If one considers a frame in which gravitational waves propagate at a constant speed, then as the light cone of matter contracts, the universe will appear to expand acausally to material observers. This is illustrated in Figure 2.

To get some feel for how this happens, and what other consequences there are, we write (13b) in terms of the Ricci curvature of $\bar{g}_{\mu\nu}$:

$$\bar{R}^{\mu\nu} = \frac{1}{2} \frac{\mu}{\bar{\mu}} \left(T^{\mu\nu} - \frac{1}{2} \bar{g}^{\mu\nu} \bar{g}_{\alpha\beta} T^{\alpha\beta} \right) + \frac{1}{2} \kappa \frac{\hat{\mu}}{\bar{\mu}} \left(\hat{T}^{\mu\nu} - \frac{1}{2} \bar{g}^{\mu\nu} \bar{g}_{\alpha\beta} \hat{T}^{\alpha\beta} \right). \quad (22)$$

The condition that the presence of matter causes convergence of \bar{g} -timelike or \bar{g} -null geodesic congruences is that $\bar{R}_{\mu\nu} V^\mu V^\nu \geq 0$ for V^μ a \bar{g} -timelike or \bar{g} -null vector. If we assume that $\hat{\psi}^\mu$ is a timelike vector (which is a reasonable cosmological assumption) and consider the special case $V^\mu = \sigma \hat{\psi}^\mu$, we find

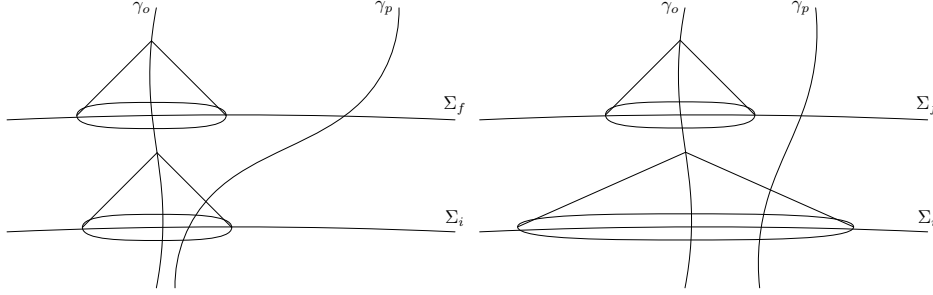


FIGURE 2. The left figure, in a frame where the velocity of matter is constant, shows the trajectory γ_p of a point p on an initial spatial hypersurface Σ_i leaving the past light cone of an observer moving on a timelike trajectory γ_o due to acausal inflation of the spacelike hypersurface between Σ_i and Σ_f . On the right, after a diffeomorphism transformation to a frame where the speed of gravitational waves is constant, we see that the same trajectory leaving the past light cone of the observer due to contraction of the observer's light cone.

the following result:

$$\begin{aligned} \bar{R}^{\mu\nu} \bar{V}_\mu \bar{V}_\nu &= \frac{1}{2} \frac{\mu}{\bar{\mu}} \sigma^2 \frac{1 + g\psi^2}{(1 + b\psi^2)^2} \left[\mathcal{T}_\psi - \frac{1}{2} (b - 2g) \psi^2 T^{\alpha\beta} \psi_\alpha \psi_\beta \right] \\ &+ \frac{1}{2} \kappa \frac{\hat{\mu}}{\bar{\mu}} \sigma^2 \left(1 + (g - b) \hat{\psi}^2 \right) \left[\hat{\mathcal{T}}_\psi - \frac{1}{2} (b - g) \hat{\psi}^2 \hat{T}(\psi, \psi) \right]. \end{aligned} \quad (23)$$

where $\hat{T} := \hat{T}^{\mu\nu} \hat{g}_{\mu\nu}$ and $T := T^{\mu\nu} g_{\mu\nu}$. In (23) we have written $\mathcal{T}_\psi = T^{\alpha\beta} \psi_\alpha \psi_\beta - \frac{1}{2} \psi^2 T$ and $\hat{\mathcal{T}}_\psi = \hat{T}^{\alpha\beta} \hat{\psi}_\alpha \hat{\psi}_\beta - \frac{1}{2} \hat{\psi}^2 \hat{T}$, both of which are non-negative if the strong energy condition is satisfied by the vector field and matter (in the matter frame), respectively. If $b > 2g$ then the second contribution from each energy-momentum tensor will be non-positive (assuming that each also satisfies the weak energy condition), and we find that a defocusing of geodesics is possible.

This implies that in the presence of a nonzero ψ_μ , the singularity theorems [15] as applied to $\bar{g}_{\mu\nu}$ are weakened. As we shall see later, this will presumably have no effect on the collapse to a black hole in the present epoch, since in the vector field models the cosmological solution requires that $\psi_\mu = 0$ when the matter energy density falls below a threshold. It is nevertheless reasonable to expect that the presence of ψ_μ , and the light cone fluctuations that result, would have an effect on the threshold of black hole formation [20].

This result is also unfortunately not so clear as far as its physical implications. Although we can speak of singularity theorems as applied to $\bar{g}_{\mu\nu}$, because the field equations that determine it are very similar to those of GR, it is the metric $\hat{g}_{\mu\nu}$ that is physically measured by the propagation of

material test bodies. Indeed, the issue of how to identify a singularity is exacerbated in these models, because one can imagine the possibility that $\hat{g}_{\mu\nu}$ is well-behaved, whereas $\bar{g}_{\mu\nu}$ and ψ_μ individually are not. The main point is that ‘ordinary’ matter can contribute to the field equations for $\bar{g}_{\mu\nu}$ as if it violates the strong energy condition. We will see exactly this happening in the following section.

5. COSMOLOGY

Implicit in the idea of a varying lightspeed is that the speed of light is changing with respect to some fixed frame of reference. If one introduces a fundamental frame for this, then it is perhaps sensible to introduce a function $c(t, x)$ to describe this variability [8, 10]. The models that we are considering are based on the idea that the speed of light can be changing with respect to the speed of gravitational disturbances, and therefore any indication of the speed of light as a function of spacetime is frame-dependent. In particular, we will see that a frame in which the speed of light is constant and the speed of gravitational disturbances is changing is connected via a diffeomorphism to a frame where the speed of gravitational disturbances is constant, and the speed of light is changing (*c.f.*, Figure 2). We will derive the quantities of interest (the local light cone, horizons, *etc.*) directly from the relevant metric, thereby avoiding any guesswork as to which ‘speed of light’ to use—the gravitational or electromagnetic [3, 4]. The *constant* c is fixed in the present universe by making measurements of the electromagnetic field.

In a homogeneous and isotropic (FRW) universe, the vector field ψ_μ has components $\psi_\mu = (c\psi_0(\tau), 0, 0, 0)$. We will begin with the metric $g_{\mu\nu}$ in comoving form

$$g_{\mu\nu}dx^\mu \otimes dx^\nu = c^2 d\tau \otimes d\tau - R(\tau)^2 \gamma_{ij} dx^i \otimes dx^j, \quad (24a)$$

and therefore

$$\hat{g}_{\mu\nu}dx^\mu \otimes dx^\nu = \hat{\Theta}^2 c^2 d\tau \otimes d\tau - R(\tau)^2 \gamma_{ij} dx^i \otimes dx^j, \quad (24b)$$

$$\bar{g}_{\mu\nu}dx^\mu \otimes dx^\nu = \bar{\Theta}^2 c^2 d\tau \otimes d\tau - R(\tau)^2 \gamma_{ij} dx^i \otimes dx^j. \quad (24c)$$

The spatial metric in spherical coordinates has the standard form

$$\gamma_{ij} = \text{diag}(1/(1 - kr^2), r^2, r^2 \sin^2 \theta), \quad (25)$$

and we have defined

$$\hat{\Theta} := \sqrt{1 + 2bX}, \quad \text{and} \quad \bar{\Theta} := \sqrt{1 + 2gX}, \quad (26)$$

where from (7) we have $X = \frac{1}{2}\psi_0^2$.

Although we begin with the choice (24a), once we have derived the field equations (which is significantly easier to do in this gauge) we will make a coordinate transformation in order to put $\hat{g}_{\mu\nu}$ in comoving form and thereby make a comparison with the standard cosmological results a simpler matter. Note that we are reversing the definitions of t and τ as used in our previous article [1].

The matter energy-momentum tensor will have a perfect fluid form:

$$\hat{T}^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) \hat{u}^\mu \hat{u}^\nu - p \hat{g}^{\mu\nu}, \quad (27)$$

where we have written the velocity field as \hat{u}^μ to emphasize that it is normalized using the metric $\hat{g}_{\mu\nu}$, so that

$$\hat{g}_{\mu\nu} \hat{u}^\mu \hat{u}^\nu = c^2. \quad (28)$$

This, we believe, is the origin of the confusion in [3, 4], in which we have been accused of making an algebraic mistake where there is none. In that work, the authors appear to have normalized the vector field as $g_{\mu\nu} \hat{u}^\mu \hat{u}^\nu = c^2$. The resulting form for $\hat{T}^{\mu\nu}$ clearly will *not* satisfy (3), and as we have seen in Section 3, will lead to inconsistent field equations since the Bianchi identities will now lead to nontrivial constraints on the matter fields.

The correct choice (28) leads to $\hat{u}^0 = 1/\hat{\Theta}$, and therefore

$$\hat{T}^{00} = \frac{\rho}{\hat{\Theta}^2}, \quad \hat{T}^{0i} = 0, \quad \hat{T}^{ij} = \frac{p}{R^2} \gamma^{ij}. \quad (29)$$

Thus equation (58) of [3] is incorrect; the first term on the right-hand side should be $\rho/\sqrt{1 + \beta\psi_0^2}$ rather than $\sqrt{1 + \beta\psi_0^2}\rho$.

The matter conservation laws (3) lead to the usual relation

$$\partial_\tau \rho + 3 \frac{\partial_\tau R}{R} \left(\rho + \frac{p}{c^2}\right) = 0. \quad (30)$$

Note that this form also holds with any choice of time coordinate $\tau \rightarrow \tau(t)$.

The components of the energy-momentum tensor for ψ are

$$T^{00} = \frac{1}{c^2} (2XV' - V), \quad T^{0i} = 0, \quad T^{ij} = V \frac{1}{R^2} \gamma^{ij}, \quad (31)$$

and using the components of the Einstein tensor

$$\bar{G}^{00} = \frac{3}{c^4 \bar{\Theta}^4} \left[\left(\frac{\partial_\tau R}{R} \right)^2 + \frac{kc^2 \bar{\Theta}^2}{R^2} \right], \quad (32a)$$

$$\bar{G}^{ij} = -\gamma^{ij} \frac{1}{c^2 R^2 \bar{\Theta}^2} \left[2 \frac{\partial_\tau^2 R}{R} + \left(\frac{\partial_\tau R}{R} \right)^2 + \frac{kc^2 \bar{\Theta}^2}{R^2} - 2 \frac{\partial_\tau R}{R} \frac{\partial_\tau \bar{\Theta}}{\bar{\Theta}} \right], \quad (32b)$$

from (13b) we have the Friedmann equations

$$\left(\frac{\partial_\tau R}{R} \right)^2 + \frac{kc^2 \bar{\Theta}^2}{R^2} = \frac{\kappa c^4}{6} \bar{\Theta}^3 \left[\frac{\rho}{\bar{\Theta}} + \frac{1}{\kappa c^2} (2XV' - V) \right], \quad (33a)$$

$$2 \frac{\partial_\tau^2 R}{R} + \left(\frac{\partial_\tau R}{R} \right)^2 + \frac{kc^2 \bar{\Theta}^2}{R^2} - 2 \frac{\partial_\tau R}{R} \frac{\partial_\tau \bar{\Theta}}{\bar{\Theta}} = -\frac{\kappa c^2}{2} \bar{\Theta} \left[\hat{\Theta} p + \frac{1}{\kappa} V \right]. \quad (33b)$$

The single remaining Proca field equation from (13a) is

$$\frac{1}{c\bar{\Theta}} \psi_0 \left[\hat{\Theta} (\bar{\Theta}^2 V' - gV) - \kappa (b - g) c^2 \rho \right] = 0. \quad (33c)$$

We now perform the coordinate transformation

$$dt = \hat{\Theta} d\tau, \quad (34)$$

and defining

$$\eta = \frac{\bar{\Theta}}{\hat{\Theta}} = \sqrt{\frac{1+2gX}{1+2bX}}, \quad (35)$$

we see that the metric $\hat{g}_{\mu\nu}$ is put into comoving form

$$\hat{g}_{\mu\nu} dx^\mu \otimes dx^\nu = c^2 dt \otimes dt - R^2(t) \gamma_{ij} dx^i \otimes dx^j, \quad (36a)$$

$$\bar{g}_{\mu\nu} dx^\mu \otimes dx^\nu = \eta^2(t) c^2 dt \otimes dt - R^2(t) \gamma_{ij} dx^i \otimes dx^j. \quad (36b)$$

Making the change of time coordinate directly on (33), (33c) is unchanged, and we find

$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{kc^2\eta^2}{R^2} = \eta^2 \frac{\kappa c^4}{6} \rho_{\text{eff}}, \quad (37a)$$

$$2\frac{\ddot{R}}{R} + \left(\frac{\dot{R}}{R}\right)^2 + \frac{kc^2\eta^2}{R^2} - 2\frac{\dot{R}}{R}\frac{\dot{\eta}}{\eta} = -\eta^2 \frac{\kappa c^2}{2} p_{\text{eff}}, \quad (37b)$$

where we will write, for example $\dot{\rho} = \partial_t \rho$. In (37), we have defined the effective energy and pressure densities as

$$\rho_{\text{eff}} = \eta \left(\rho + \frac{1}{\kappa c^2} \hat{\Theta} (2XV' - V) \right), \quad p_{\text{eff}} = \frac{1}{\eta} \left(p + \frac{1}{\kappa \hat{\Theta}} V \right). \quad (38)$$

The reason for making these definitions is that (37a) has exactly the form of the Friedmann equations for the metric $\bar{g}_{\mu\nu}$ in (36b), and therefore these effective energy and momentum densities will also satisfy the conservation laws (30).

Some careful observations are in order. The function $R(t)$ is determined from (37a) in which an (unusual) effective energy density appears. It also formally appears to be a ‘varying constants’ model with $c(t) = \eta(t)c$ and $G(t) = \eta^2(t)G$ coupled to ρ_{eff} . We could equally well expand ρ_{eff} to find a varying constants form with

$$c(t) = \eta c, \quad G(t) = \eta^3 G, \quad \Lambda(t) = \frac{1}{2} \bar{\Theta} (2XV' - V), \quad (39)$$

or, if we interpret (37a) as a varying constants theory written in a non-comoving coordinate system with metric of the form (36b), we would have

$$c(t) = c, \quad G(t) = \eta G, \quad \Lambda(t) = \frac{1}{2} \bar{\Theta} (2XV' - V), \quad (40)$$

and, as we will see in Section 5.2, once the vector field equation (33c) is solved, the resulting system need not take on an FRW form at all. Clearly any such identification is ambiguous.

Note though, that the resulting function $R(t)$ appears in (36a), which is written in comoving coordinates, and therefore the speed of light is constant. This emphasizes that having a ‘varying speed of light’ is a frame-dependent statement. In a frame where the speed of matter propagation (including electromagnetic fields) is constant, the speed of gravitational waves will be changing. In a frame where the speed of gravitational waves is constant, the speed of matter propagation will be changing. This, of course, is as it

should be, since we have not introduced any nondynamical preferred frame into our model.

In the following we will specialize to a model where the vector field potential is a simple mass term:

$$V = m^2 X, \quad V' = m^2. \quad (41)$$

In this case (38) becomes

$$\rho_{\text{eff}} = \eta \left(\rho + (b - g) \rho_{pt} \hat{\Theta} X \right), \quad p_{\text{eff}} = \frac{1}{\eta} \left(p + c^2 (b - g) \rho_{pt} \frac{X}{\hat{\Theta}} \right), \quad (42)$$

and we find for later use that

$$\rho_{\text{eff}} + \frac{3}{c^2} p_{\text{eff}} = \eta \rho + \frac{3}{c^2 \eta} p + (b - g) \rho_{pt} \left(\bar{\Theta} X + \frac{3}{X \bar{\Theta}} \right). \quad (43)$$

The nontrivial solution ($\psi_0 \neq 0$) of the field equation (33c) leads to

$$\rho = \rho_{pt} \hat{\Theta} (1 + gX), \quad (44)$$

where

$$\rho_{pt} = \frac{m^2}{\kappa c^2 (b - g)}, \quad H_{pt} = \sqrt{\frac{c^2 m^2}{6(b - g)}}, \quad (45)$$

are the density at which $\psi_0^2 = 0$ is reached, and the inverse Hubble time at which this occurs (assuming that $k = 0$).

We can now write the acceleration parameter as observed by material observers from (37) as

$$\hat{q} = -\frac{\ddot{R}}{H^2 R} = \frac{\kappa c^4}{12} \frac{\eta^2}{H^2} \left(\rho_{\text{eff}} + \frac{3}{c^2} p_{\text{eff}} \right) - \frac{\dot{\eta}}{H \eta}, \quad (46)$$

where we have defined the Hubble function $H = \dot{R}/R$. Taking a derivative of the definition (35), relating the result to $\dot{\rho}$ using the derivative of (44), and removing $\dot{\rho}$ using (30), we find

$$-\frac{\dot{\eta}}{H \eta} = -3 \frac{(b - g)}{\rho_{pt} \bar{\Theta}^2 \hat{\Theta} (g + b + 3bgX)} \left(\rho + \frac{p}{c^2} \right). \quad (47)$$

We will return to this shortly.

5.1. The very early universe. For very short times following the initial singularity, we expect that ψ_0 is large, and if we assume that $gX \gg 1$ and $bX \gg 1$, then from (44) we find that

$$\rho = \rho_{pt} \sqrt{2bg} X^{3/2}. \quad (48)$$

This results in the Friedmann equation

$$\left(\frac{\dot{R}}{R} \right)^2 + \frac{k \bar{c}^2}{R^2} = \frac{\bar{\kappa} \bar{c}^4}{6} \rho, \quad (49)$$

where

$$\bar{c} = c\sqrt{\frac{g}{b}}, \quad \bar{G} = G\sqrt{\frac{g}{b}}, \quad \bar{\kappa} = \frac{16\pi\bar{G}}{\bar{c}^4}. \quad (50)$$

Although the behaviour of the solutions to (49) are well-known [21], it is worth pointing out that the ‘effective’ constants \bar{c} and \bar{G} are *not* interpretable as the effective speed of light and gravitational constant, rather they are effective constants that dictate how the gravitational field reacts to the presence of matter. Matter fields continue to propagate with speed c consistent with (36a). It is the gravitational field perturbations that propagate with speed \bar{c} , which is the justification for the notation.

During this phase there is clearly no inflation, but the horizon scales of the gravitational field and matter fields are related by:

$$\bar{d}_H(t) = \frac{\bar{c}}{c}\hat{d}_H(t) = \sqrt{\frac{g}{b}}\hat{d}_H(t), \quad \text{where} \quad \hat{d}_H(t) = cR(t) \int_0^t \frac{ds}{R(s)}, \quad (51)$$

with a similar definition for $\bar{d}_H(t)$ using the metric (36b). Because we have $g < b$ we expect that not only is the speed of gravitational disturbances slower than that of matter, but also that the coupling between matter and the gravitational sector is also lessened.

What we have here is very close to what was originally envisaged by one of us in [7]. This is part of the motivation for including the $g \neq 0$ possibility, the other is that the approach to the initial singularity in this phase follows the same path as in ordinary GR+matter models, with a re-interpretation of the parameters. In this case we have a model that interpolates between this initial period where $\bar{c} > c$ and the later universe where $\bar{c} = c$. We turn to this next.

5.2. Inflation and light cone contraction. As ψ_0 decreases towards the point where $gX \sim 1$ the solution found in Section 5.1 will no longer be a good approximation. If we now consider the solution when $gX \ll 1$, from (44) we have

$$\hat{\Theta} = \frac{\rho}{\rho_{pt}}, \quad \text{or} \quad X = \frac{1}{2b} \left[\left(\frac{\rho}{\rho_{pt}} \right)^2 - 1 \right], \quad (52)$$

and the Friedmann equation (37a) becomes

$$\left(\frac{\dot{R}}{R} \right)^2 + \frac{kc^2\eta^2}{R^2} \left(\frac{\rho_{pt}}{\rho} \right)^2 = \frac{\kappa c^4}{12} \rho_{pt} \left[1 + \left(\frac{\rho_{pt}}{\rho} \right)^2 \right]. \quad (53)$$

In this limit

$$\rho_{\text{eff}} + \frac{3}{c^2} p_{\text{eff}} = \frac{1}{\rho_{pt}} \left[\rho \left(\rho + \frac{3}{c^2} p \right) + \rho^2 - \rho_{pt}^2 \right], \quad (54)$$

which is greater than zero if the strong energy condition is satisfied, since $\rho \geq \rho_{pt}$, and (46) reduces to

$$\hat{q} = \frac{\kappa c^4 \rho_{pt}}{12H^2} \left[\frac{1}{\rho} \left(\rho + \frac{3}{c^2} p \right) + 1 - \left(\frac{\rho_{pt}}{\rho} \right)^2 \right] - \frac{3}{\rho} \left(\rho + \frac{p}{c^2} \right). \quad (55)$$

Since we expect that H^2 is large in the early universe (we can arrange that $\rho_{pt} \ll \rho_c$ where $\rho_c = 12H^2/(\kappa c^4)$), it is clear from (55) that even if matter satisfies the strong energy condition, the final term will dominate and $\hat{q} < 0$ (unless, perhaps, the weak energy condition is also violated). This is the expansion of the universe as seen by material observers. The acceleration of the gravitational geometry \bar{q} would lack the final term and therefore $\bar{q} > 0$.

That we get inflation was demonstrated previously [1], where an exact solution for $k = 0$ and $g = 0$ was found. Although we discovered that we could not get enough expansion to solve the horizon problem with pure radiation, a slowly rolling scalar field could provide the necessary negative pressure. The role that the extra structure of our model plays is that the fine-tuning that is required in a simple scalar field, potential-driven model is alleviated.

The flatness problem requires a bit more explanation. Dividing (37a) by H^2 and defining

$$\bar{\epsilon} = \frac{kc^2\eta^2}{(\dot{R})^2}, \quad (56)$$

we find a differential equation that $\bar{\epsilon}$ satisfies by taking a derivative and using (37) to give

$$\dot{\bar{\epsilon}} = \frac{\kappa c^4 \eta^2}{6H} \bar{\epsilon} \left(\rho_{\text{eff}} + \frac{3}{c^2} p_{\text{eff}} \right). \quad (57)$$

Therefore, since $\bar{\epsilon} > 0$ and $H > 0$ in the early universe, the only way for $\bar{\epsilon} = 0$ to be an attractor for (37a) is for $\rho_{\text{eff}} + \frac{3}{c^2} p_{\text{eff}} < 0$ at least for part of the history of the universe. What is not so obvious is whether the quantity $\bar{\epsilon}$ as defined in (56) is of physical relevance.

Following [22, 23] the quantity of geometrical importance is the 3-curvature of the spacelike slices: $6k/R^2$, which suggests that the physically meaningful quantity to examine would be

$$\hat{\epsilon} = \frac{kc^2}{(\dot{R})^2}, \quad (58)$$

which has the equation of motion

$$\dot{\hat{\epsilon}} = 2\hat{\epsilon}\hat{q}. \quad (59)$$

Another way of stating this is that the curvature radius defines Ω through [22, 23]:

$$R_{\text{curv}} = \frac{R}{|k|^{1/2}} = \frac{c}{H(|\Omega - 1|)^{1/2}}, \quad (60)$$

and so $\hat{\epsilon} = |\Omega - 1|$. Since we found from (55) that $\hat{q} < 0$ in the early universe, clearly $\hat{\epsilon} = 0$ is an attractor for (37a), and since it is most-likely the quantity of physical importance for matter physics, we can also claim to have solved the flatness problem once the horizon problem is solved.

Some comments on the remarks made in [3] on the flatness problem are in order. If we look at the acceleration of the universe from the point of view of gravitational ‘observers’ only, then the universe will not appear to inflate and the flatness problem is not solved. This appears to be the point of view advocated implicitly in [3]. The point we are making here is that the geometry that we observe as material bodies is that described by $\hat{g}_{\mu\nu}$, and as far as material observers are concerned the universe will appear to inflate. As it does so, the spatial curvature that would be measured by such observers necessarily increases. That is, the curvature radius sets the length scale of the size of spatial separations on which the effects of spatial curvature become important. That, of course, is the rationale behind the statement that “the horizon and flatness problems are geometrically linked”.

6. CONCLUSIONS

One of our purposes here was to give a more complete description of the model than was given in [1]. In doing so we showed that the universe generically accelerates ($\hat{q} < 0$) during some period in the early universe, and that in the same period the physical importance of spatial curvature diminishes ($|\Omega - 1|$ is decreasing). This can occur even when the matter fields satisfy the strong energy condition. This conclusion is the opposite of that which appeared recently [3, 4], who take a somewhat different point of view on the interpretation of the metrics appearing in these theories. Nevertheless, we have demonstrated conclusively that the claim appearing in [3] that we had made an algebraic error in [1] is false.

The model that we have considered generalizes that which appeared in [1] in a way that more closely follows the scenario discussed in [7]. In the very early universe, matter and gravitational fields propagate with different and approximately constant velocities. During a period during which the matter light cone, originally much larger than the light cone of gravity, contracts, material observers will see an acausal expansion of the universe similar to inflation. Because the light cone of gravity does not undergo the same contraction, we expect there to be an observable difference in the scalar versus tensor contributions to the cosmic microwave background anisotropies.

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